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Existence Theorems of Continuous Social Aggregation for Infinite Discrete Alternatives

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Abstract

This paper considers the infinite alternative case to prove the existence of continuous social welfare aggregation that is anonymous and respects the unanimity. It clarifies the controversy between Chichilnisky (1982, QJE) and Huang for their contradictory results for the continuum case. Compared to their topological frameworks, the infinite alternative case is easier to understand and pinpoint their difference.

1 Introduction

Built on Arrow's (1951) framework, Chichilnisky (1982) proves that for a continuum alternative space X, there exists no continuous social welfare function on profiles of preferences that is anonymous and respects unanimity. These theorems have resulted in two of the most-cited works in the area of social choice theories.

Huang (2009) proves the existence of continuous social utility maps that are anonymous and respect unanimity, contrary to Chichilnisky's (1982) impossibility theorem. Huang introduces the notion of singularity of social aggregations and notes that in Chichilnisky's framework the singularity that relates to the set of zero preference vectors is not properly treated. On the other hand, Huang (2014) reexamines Arrow's paradox and proposes the extent principle to revise the form of Arrow's independence. He shows Arrow's framework excludes a singularity such as cyclic social preference orderings. To tackle the behavior of continuous variation in preferences on X collectively while dealing carefully with singularity, Huang (2004, 2009, 2014) uses the technical language of topology, which reduces the tractability and applicability of his findings.

For purposes of helping explain Huang's (2009) existence theorem, we consider a simpler setting, where X is an infinite discrete set $\{x_1, x_2, x_3, ...\}$. Under this joint setting between discrete and continuum cases, we reexamine the existence of continuous social aggregation in a manner analogous to Chichilnisky (1982).

The rest of the paper is organized as follows. Section 2 introduces the analysis framework, and Section 3 proves the existence theorem.

2 The framework

Consider $X = \{x_i; i \in \mathbb{Z}^+\}$ as an alternative space, i.e., X consists of infinite discrete alternatives, where $\mathbb{Z}^+$ is the set of natural numbers. The conventional topology of X is generated by the base $\mathfrak{B} = \{B_i; i \in \mathbb{Z}^+\}$, where $B_i = \{x_i, x_{i+1}, x_{i+2}, ...\}$. A *preference* p on X is a transitive binary order over any pairs of X, i.e., (i) $\forall x, y \in X$, we have $x \succeq y$, or $y \succeq x$, or both; (ii) $\forall x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Given a preference p, "$x \succeq y$ (in p)" indicates "x is at least as good as y in the preference p." The *strict preference* $\succ$ is defined by "$x \succ y \Leftrightarrow x \succeq y$ but not $y \succeq x$" (that is, x is preferred to y). The *indifference preference* $\sim$ is defined by "$x \sim y \Leftrightarrow x \succeq y$ and $y \succeq x$" (that is, x is indifferent to y).

A preference p on X is called *regular*, if there is no pair of (distinct) alternatives $x, y \in X$ with $x \sim y$ in p. Otherwise, p is called *singular.* Given any V and W in X, "$V \succ W$ in p" means "$v \succ w$ in p" for any v in V and w in W. The totality of all preferences on X is denoted by P, while $P^* \subset P$ denotes the set of all regular preferences on X. Let $p \in P$, given $x \in X$ we consider the superior set $U_x(p) \equiv \{y \in X; y \succeq x \text{ in } p\}$, the inferior set $L_x(p) \equiv \{y \in X; y \preceq x \text{ in } p\}$ and the indifferent set $I_x(p) \equiv \{y \in X; y \sim x \text{ in } p\}$.

In an economy with N individuals, let p_α be the preference of the individual $\alpha = 1, 2, ..., N$; let $\mathbf{p} = (p_1, p_2, ..., p_N)$ denote a profile of individual preferences; and let P^N denote the set of all profiles. Let $\mathfrak{F}(X)$ be the space of real valued functions defined on X. A *social utility map* $\mathcal{U}$ is a map,

$$\mathcal{U} : P^N \to \mathfrak{F}(X), \tag{2.1}$$

assigning to each profile a real valued function $u \in \mathfrak{F}(X)$, which we may call a *social utility function* on X. A *social welfare function* is a map, $F : P^N \to P$, which assigns to each profile a social preference.

Analogous to local setting of preference vector fields, defined by Antonelli (Debreu 1972) and developed by Chichilnisky (1982), *a local preference* on infinite discrete X is an assignment, $v : x_i \to v(x_i) \in \{-1, 0, +1\}$, assigning to each alternative x_i a number -1, 0 or +1, which respectively indicates "$x \succ, \sim$ or $\prec x_{i+1}$".

Let P_c denote the totality of local preferences. A local preference v provides preference order between x_i and x_{i+1}, but may not do so for x_i and x_{i+2}. For example, when $v(x_i) = +1, v(x_{i+1}) = -1$, we cannot judge whether x_{i+2} is preferred to x_i. In this sense, we call it "local." Comparatively, a preference p in P is called a *global preference* on X, as it defines orders over *all* pairs x, y of X.

Given an abstract set G, a topology defines a consistent way of "convergence" among element in G, i.e. how "a_t converges to a, where $a_t, a \in G$." It means how elements of G "vary continuously" in rigorous mthematical terms. We will define various ways of convergence on preference spaces P and P_c by introducing topologies as follows:

Definition 2.1. *Let the topological spaces $(P, \Im)$, $(P_c, \Im_c)$ and $(P, \Im_0)$ be defined as follows.*

1. *p_t converge to p_0 in the global preference topology $\Im$ (usually denoted by $p_t \to p_0$ in $\Im$) if and only if for any finite set A in X, there exists a number T such that $\forall t > T$,*

$$p_t|_A = p_0|_A. \tag{2.2}$$

The last formula means: $\forall x, y \in A, x \succeq y$ in p_t iff $x \succeq y$ in p_0.

2. *v_t converges to v_0 in the local preference topology $\Im_c$ (usually denoted by $v_t \to v_0$ in $\Im_c$) if and only if for any finite set A in X, there exists a number T such that $\forall t > T$, $v_t|_A = v_0|_A$, where "$|_A$" means the function restriction, i.e. $\forall x \in A$, $v_t(x) = v_0(x)$.*

3. *p_t converges to p_0 in the social preference topology $\Im_0$ (usually denoted by $p_t \to p_0$ in $\Im_0$) if and only if for any pair of disjoint finite sets V, W in X with $V \succ W$ in p_0, there exists a number T such that $V \succ W$ in p_t $\forall t > T$.*

We note that the global preference topology $\Im$ is stronger than the social preference topology.

Proposition 1.

$$p_t \to p \text{ in } \Im \Rightarrow p_t \to p \text{ in } \Im_0. \tag{2.3}$$

The converse is not true unless p is regular.

Proof. Given any two finite sets V and W in X with $V \prec W$ in p, we choose a finite set A such that $A \supset V \cup W$. By $p_t \to p$ in $\Im$, $\exists$ T such that $p_t|_A = p|_A$, $\forall t > T$. Hence, $V \prec W$ in p_t, $\forall t > T$. Thus $p_t \to p$ in $\Im_0$. As for the converse, consider two preferences q_1 and q_2 defined by

$$\begin{aligned} q_1: &\quad x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec ... \\ q_2: &\quad x_2 \prec x_1 \prec x_3 \prec x_4 \prec x_5 \prec ... \end{aligned}$$

both monotone after x_3. We define a sequence of preferences $\{p_t\} = \{q_1, q_2, q_1, q_2, q_1, q_2, ...\}$ and $p : x_1 \sim x_2 \prec x_3 \prec x_4 \prec x_5 \prec ....$ Clearly, $p_t \to p$ in $\Im_0$. In fact, if V and W are two non-empty sets in X with $V \prec W$ in p, then $\forall x_j$ in W, we see that $j \geq 3$ since $V \neq \emptyset$. By monotonicity of q_1 and q_2 after $j \geq 3$, it holds that $V \prec W$ in p_t. On the other hand, it is evident that P_t does not converge to p in $\Im$. Hence, the converse of (2.3) is not true.

The proof of Proposition 1 tells us the basic difference between $\Im$ and $\Im_0$ is focused on singularities. We notice that the converse of (2.3) is not true only when p is singular. If p is regular, it becomes that

$$p_t \to p \text{ in } \Im \Leftrightarrow p_t \to p \text{ in } \Im_0. \qquad \blacksquare$$

The map ψ defined in the following provided a link between P and P_c.

Definition 2.2. *The localization map $\psi : P \to P_c$ is defined by*

$$\psi(p)(x_i) = \begin{cases} 1 & \text{if } \ x_{i+1} \succ x_i \\ 0 & \text{if } \ x_{i+1} \sim x_i \\ -1 & \text{if } \ x_{i+1} \prec x_i \end{cases}$$

Remark 1. *The map ψ is surjective and many to one. For example, consider*

$$p_1: \quad x_1 \prec x_2 \succ x_3 \sim x_4 \sim x_5 \sim x_6 \sim \cdots, \ \textit{but } x_1 \prec x_3,$$
$$p_2: \quad x_1 \prec x_2 \succ x_3 \sim x_4 \sim x_5 \sim x_6 \sim \cdots, \ \textit{but } x_1 \succ x_3,$$
$$p_3: \quad x_1 \prec x_2 \succ x_3 \sim x_4 \sim x_5 \sim x_6 \sim \cdots, \ \textit{but } x_1 \sim x_3.$$

Then $\psi(p_k) =$ same $v_0 \in P_c$, $\forall k = 1, 2, 3,$ where $v_0(x_1) = +1$, $v_0(x_2) = -1$, $v_0(x_3) = 0 = v_0(x_4) = \cdots$.

Proposition 2. *The topology $\Im$ of the global preference space P is equivalent to the topology $\Im_c$ of the local preference space P_c under the localization map ψ, in the sense that for any set $\tau \subset P_c$.*

$$\tau \textit{ is open in } (P_c, \Im_c) \textit{ iff } \psi^{-1}(\tau) \textit{ is open in } (P, \Im).$$

And furthermore, ψ is an open map, i.e., for any σ open in $(P, \Im)$, $\psi(\sigma)$ is open in $(P_c, \Im_c)$.[1]

Proof. See the Appendix. ∎

3 Existence theorems

Definition 3.1. *The* cardinality-forgetting *map $\pi : \mathfrak{F}(X) \to P$ is defined by $p := \pi(u) \in P$, for any $u \in \mathfrak{F}(X)$ such that*

$$x \succsim y \textit{ in } p \textit{ iff } u(x) \geq u(y), \forall x, y \in X.$$

We call u a utility function on X defining *the global preference $p \in P$, or call p the preference* corresponding *to utility function u.*

Proposition 3. *Let $p_t = \pi(u_t)$, $p = \pi(u)$ where $u_t, u \in \mathfrak{F}(X)$. Then*

$$u_t \to u \textit{ uniformly on } X \Rightarrow p_t \to p \in \Im_0,$$

but under the same hypothesis, p_t may not converge to p in $\Im$.

[1]A topology of an abstract set G can be defined either by the notion of "openness" or "convergence." If by the former, we introduce a family $\theta \equiv \{u_\alpha; \alpha \in I\}$ of subsets u_α of G in which each u_α is called an open set, such that (i) the empty set ϕ and the entire set G are open (i.e., contained in θ); (ii) any union of open sets is open; (iii) any finite intersection of open sets is open. (B) If a topology of G is defined by the notion of "convergence" then it defines "openness" in the sense that $U \subset G$ is said to be *open* in G if and only if for any convergent sequence

$$x_t \to x_0 \text{ in } G, \tag{2.4}$$

where $x_0 \in U$, there exists T such that $x_t \in U$, $\forall t > T$.

Proof. (i) Given finite sets $V, W \subset X$ with $V \prec W$ in p, it holds that

$$u(v) < u(w), \forall v \in V, w \in W,$$

since $p = \pi(u)$. To show that $p_t \to p$ in $\Im_0$, it suffices to show that $\exists\, T$ such that $V \prec W$ in $p_t \forall t > T$. In fact, we may choose ε so small that $u(v) < u(w) - 2\varepsilon$. By $u_t \to \mathfrak{F}(X)$, $\exists\, T$ such that $\forall v \in V, w \in W$,

$$|u_t(v) - u(v)| < \varepsilon \text{ and } |u_t(w) - u(w)| < \varepsilon, \forall t > T.$$

Hence, $u_t(v) < u(v) + \varepsilon < (u(w) - 2\varepsilon) + \varepsilon = u(w) - \varepsilon < u_t(w)$. As $p_t = \pi(u_t)$, we have $V \prec W$ in $p_t, \forall t > T$, as required.

(ii) To claim p_t may not converge to p in $\Im$, we let

$$u(x_j) = \begin{cases} 1 & for \quad j = 1,2 \\ j & for \quad j > 2 \end{cases} \quad and \quad u_t(x_j) = \begin{cases} 1 - 1/t & for \quad j = 1 \\ 1 + 1/t & for \quad j = 2, \\ j & for \quad j > 2. \end{cases}$$

Then $u_t \to u$ uniformly on X. Now let $p = \pi(u)$, $p_t = \pi(u_t)$. We see that $x_1 \prec x_2$ in p_t $\forall t$. However, $x_1 \sim x_2$ in p. Thus, p_t does not converge to p in $\Im$. ■

Theorem A. *Given an infinite discrete alternative set X, let $\mathfrak{F}(X)$ denote the space of all of the real-valued functions on X, and P denote the totality of preferences on X. If P is equipped with the global preference topology $\Im$, then there exist social utility maps*

$$U : P^N \to \mathfrak{F}(X)$$

where P^N is the space of N-profiles of preferences equipped with product topology of $\Im$ on P, and U satisfies the following properties:

1. *Continuity: For any sequence of profile $\mathbf{p}_t$ in P^N,*

$$\mathbf{p}_t \to \mathbf{p} \text{ in } \Im^N \Rightarrow U(\mathbf{p}_t) \to U(\mathbf{p}) \text{ uniformly on } X.$$

2. *Anonymity: $U(p_1, ..., p_i, ...p_j, ..., p_N) = U(p_1, ..., p_j..., p_i..., p_N)$ $\forall i, j \in \mathbb{Z}^+$, where p_i and p_j interchange their positions.*

3. *Respecting Unanimity: If all N individuals have a common preference $p \in P$, then the social utility $U(p, p, ..., p)$ defines the preference p, i.e. $\pi(U(p, p, ..., p)) = p$.*

Proof. **Step 1** Consider the sequence of maps, $P^N \xrightarrow{\eta^N} \mathfrak{F}(X)^N \xrightarrow{G_k} \mathfrak{F}(X)$. Here $\eta : P \longrightarrow \mathfrak{F}(X)$ maps a preference p to a utility function u_p defined by

$$u_p(x) = \mu(L_x(p)) \equiv \frac{1}{2^{j_1}} + \frac{1}{2^{j_2}} + \ldots > 0, \forall x \in X, \tag{3.1}$$

where the inferior set

$$L_x(p) = \{x_{j_1}, x_{j_2}, \ldots\} \text{ with } j_1 < j_2 < \ldots. \tag{3.2}$$

And G_k is a symmetric function ($k = 1, \ldots, N$) defined by

$$G_k(u_1, u_2, \ldots, u_N) = \sum_{I_k} \exp(u_{i_1}) \exp(u_{i_2}) \cdots \exp(u_{i_k}), \tag{3.3}$$

where $u_i \in \mathfrak{F}(X)$ and I_k indicate the set of all combinations $\{i_1, i_2, \cdots, i_k\}$ of $\{1, 2, \cdots, N\}$

Step 2 Given $p \in P$, the following three statements are equivalent:

$$x \succeq y \text{ in } p \Leftrightarrow L_x(p) \supseteq L_y(p) \Leftrightarrow u_p(x) \geq u_p(y). \tag{3.4}$$

Let the social utility map U be defined by $U = G_k \circ \eta^N$. Clearly, U satisfies anonymity since G is a symmetric function. We show that U satisfies strong Pareto principle. Given $x \succeq y$ in each individual preference p_α, $\forall \alpha \in \{1, 2, \cdots, N\}$, we have $u_{p_\alpha}(x) \geq u_{p_\alpha}(y)$ for any α by (3.4). Evidently, $G_k(u_{p_1}(x), \cdots, u_{p_N}(x)) \geq G_k(u_{p_1}(y), \cdots, u_{p_N}(y))$, which yields that $(G_k \circ \eta^N(\mathbf{p}))(x) \geq (G_k \circ \eta^N(\mathbf{p}))(y)$, where $\mathbf{p} = (p_1, ..., p_N)$. Thus $U(\mathbf{p})(x) \geq U(\mathbf{p})(y)$.

Step 3 It remains to show the continuity of U. Claim that $\eta : (P, \Im) \to \mathfrak{F}(X)$ is continuous, i.e.,

$$p_t \to p \text{ in } \Im \Rightarrow u_t \to u \text{ uniformly},$$

where $u_t = \eta(p_t) = u_{p_t}$, $u = \eta(p) = u_p$ defined by (3.1) and (3.2). For any $\varepsilon_1 > 0$, choose a finite integer δ such that

$$\delta > \frac{-\ln \varepsilon_1}{\ln 2} + 1. \tag{3.5}$$

Take $A \equiv \{x_1, x_2, ..., x_\delta\} \subset X$. Given x in X, let

$$L_x(p_t) = B_t \cup C_t, \text{ where } B_t = L_x(P_t) \cap A, \ C_t \subset X - A.$$

Similarly, let $L_x(p) = B \cup C$, where $B = L_x(P) \cap A$ and $C \subset X - A$. Since there exists

T such that $p_t|_A = p|_A$ for any $t > T$, we have $B_t = B$ for any $t > T$. However,

$$\mu(C_t) \leq \mu(X - A) \text{ and } \mu(C) \leq \mu(X - A).$$

It is clear that $\mu(X - A) = 2^{-(\delta+1)} + 2^{-(\delta+2)} + \cdots = 2^{-\delta}$. By equation (3.5), we obtain that for all $t > T$,

$$\begin{aligned} |\mu(L_x(p_t)) - \mu(L_x(p))| &= |\mu(C_t) - \mu(C)| \leq \mu(C_t) + \mu(C) \\ &\leq 2 \cdot \mu(X - A) \leq \frac{1}{2^{\delta-1}} < \varepsilon_1. \end{aligned} \tag{3.6}$$

Thus, $\mu(L_x(p_t)) \to \mu(L_x(p))$ and $u_t \to u$ pointwisely. This convergence is also uniform because the bound of (3.6) is independent of x. Therefore, η is continuous. By the continuity of the product map and the symmetric function G_k, we have $U = G_k \circ \eta^N$ is continuous. This complete the proof. ■

We note that the measure μ in (3.1) is a nonnegative measure defined on X and that the value of G_k in (3.3) depends on the choice of k. A different measure of X and different choice of k may induce various social utility maps U that assign different social utility functions to a given individual preferences profiles. In other words, there exist many different social utility maps which satisfy the given rational principles.

Combining Theorem A and Proposition 3, we obtain the existence of rational social welfare functions as follows.

Theorem B. *Given X, an infinite discrete set of alternatives, and P, the totality of preferences on X equipped with the global preference topology $\Im$. If we replace the topology $\Im$ of P by zero order topology $\Im_0$, when social preferences are considered, then there exist continuous social welfare functions,*

$$F : (P^N, \Im^N) \to (P, \Im_0),$$

which is anonymous and respects unanimity.

Proof. Let $F = \pi \circ U$, where U be the continuous utility maps given in Theorem A. Proposition 3 implies the cardinality-forgetting map, $\pi : \mathfrak{F}(X) \to (P, \Im_0)$, is continuous. Therefore, F is continuous. The requirements of anonymity and unanimity are clearly satisfied. ■

Definition 3.2. *Let 2^X denote the power set of X; that is, the set of all subsets of X. A map $C : P^N \to 2^X$ is called a choice map.*

Definition 3.3. *A choice map $C : P^N \to 2^X$, where P is equipped with the global preference* *topology $\Im$, is called continuous if*

$$x_t \in C(\mathbf{p}_t) \Rightarrow \lim_{t\to\infty} x_t \in C(\mathbf{p}),$$

wherever $\mathbf{p}_t$ converges to $\mathbf{p}$ in $\Im^N$.

Definition 3.4. *A choice map C respects unanimity if*

$$M(p_i) = \text{ same } Z \neq \phi, \forall i = 1, \cdots, N \Rightarrow C(\mathbf{p}) = Z,$$

where $M(p)$ is defined $\{x \in X; x \succeq y \text{ in } p, \forall y \in X\}$. And C is called anonymous if

$$C(p_{i_1}, p_{i_2}, \cdots, p_{i_N}) = C(p_1, p_2, \cdots, p_N),$$

for any permutation $(i_1, i_2, \cdot, i_N)$ of $(1, 2, \cdots, N)$.

Theorem C. *Given X, an infinite discrete set of alternatives, and P, the totality of preferences on X equipped with the global preference topology $\Im$. There exist continuous choice maps $C : P^N \to 2^X$ which are anonymous and respect unanimity.*

Proof. For a utility function u on X, define $S(u) = \{x \in X \ ; \ u(x) = u_0\} \subset X$, where $u_0 := \limsup\{u(y), y \in X\}$. Note that $S(u)$ may or may not be empty as X is infinite. Define

$$C(\mathbf{p}) = S(U(\mathbf{p})), \forall \mathbf{p} = (p_1, \cdots, p_N) \in P^N,$$

where U is the social utility map given in Theorem A. Since U is continuous, anonymous and respect unanimity, those requirements are statisfied. ■

Appendix: A proof of proposition 2

Step 1 Using (2.4), we set up a criterion for a set open in P. Claim that $\forall \sigma \subset P$, σ is open in P if and only if $\forall p_0 \in \sigma$, there corresponds a finite set A in X such that

$$N_A(P_0) \subset \sigma, \tag{a}$$

where $N_A(p_0) \equiv \{p \in P; p|_A = p_0|_A\}$. First we assume $\sigma \subset P$ is open in $\Im$. Suppose $\exists\ p_0 \in \sigma$ such that for any finite set $A \subset X$, $N_A(p_0) \not\subseteq \sigma$. For any $t < \infty$, let $A_t \equiv \{x_1, x_2, \cdots, x_t\}$. We select $p_t \in N_{A_t}(p_0) - \sigma \neq \phi$. Clearly, $p_t \to p_0$ in $\Im$. (In fact, $\forall A \subset X$, a finite subset, let t_0 be such that $A_{t_0} \supset A$, then $p_t|_A = p_0|_A$ for any

$t > t_0$ because $p_t \in N_{A_{t_0}}(p_0) \subset N_{A_t}(p_0) \subset N_A(p_0)$.) By σ open in P, we have $p_t \in \sigma$, for t large enough. This contradicts to $p_t \in N_{A_t}(p_0) - \sigma$, for any $t < \infty$. Thus (a) is satisfied.

The converse is proved as follows. Given $\sigma \subset P$ satisfying (a), we will show that σ is open in P in the sense of (2.4), i.e. $\forall p_t \to p_0 \in \sigma$ in $\Im$, $\exists\ T$ such that $p_t \in \sigma, \forall t > T$.

Since by (a), $\exists$ some $A \subset X$ such that $N_A(p_0) \subset \sigma$. By (2.2) in Definition (2.1), $\exists\ T$ such that $p_t|_A = p_0|_A, \forall t > T$. This implies $p_t \in N_A(p_0) \subset \sigma, \forall t > T$. So σ is open in P in the sense of (2.4).

Step 2 Claim that $\forall \tau \subset P_c$, τ is open if and only if $\forall v_0 \in \tau$, there exists a finite set A in X such that $M_A(v_0) \subset \tau$, where $M_A(v_0) = \{v \in P_c; v|_A = v_0|_A\}$. The proof is similar as in Step 1.

Step 3 Step 1 simply says that all sets of the form $N_A(p_0)$ in P constitute a "basis" of topology $\Im$. It is clear that we may assume without loss of generality that A is an sequence from x_1 to x_m, i.e. $A = \{x_1, x_2, x_3, \cdots, x_m\}$ for some m.

Step 4 Similarly, all sets of the form $M_A(v_0)$ in P_c constitute a basis of topology $\Im_c$, where A is assumed to be a sequence from x_1 to some x_m without loss of generality.

Step 5 To claim that $(P, \Im)$ and $(P_c, \Im_c)$ are equivalent under ψ, it suffices to show

$$\psi(N_A(p_0)) = M_{A'}(v_0), \tag{b}$$

where $A = \{x_1, x_2, \cdots, x_{m-1}, x_m\}$, $A^{'} = \{x_1, x_2, \cdots, x_{m-1}\}$, and $\psi(p_0) = v_0$. It means that ψ is a continuous map and is an open map. However, we note that $\psi^{-1}(M_{A'}(v_0)) \supsetneqq N_A(p_0)$.

Step 6 To show (b), we first check "$\subset$": For $p \in N_A(p_0)$, $p|_A = p_0|_A$ and $\psi(p)|_{A'} = \psi(p_0)|_{A'} = v_0|_{A'}$. Hence, $\psi(p) \in M_{A'}(v_0)$. Now we claim "$\supset$": Given $v \in M_{A'}(v_0)$, we have to construct $p \in N_A(p_0)$ such that $\psi(p) = v$. We define an utility function f on X by $f(x_i) =$ number of $\{x_j; x_j \prec x_i \text{ in } p_0 \text{ and } 1 \le j \le m\}$ for $i = 1, 2, \cdots, m$, and

$$f(x_{m+k}) = f(x_m) + v_m + v_{m+1} + \cdots + v_{m+k-1}, \text{ for } k \ge 1.$$

Finally, we define $p = \pi(f)$, that is, p is the preference corresponding to f. It is clear that $p|_A = p_0|_A$ and $\psi(p) = v$, as required.

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